Solving Polynomial Inequalities

Exercises

1. Solve each of the following polynomial inequalities using a graphical approach, $x \in \mathbb{R}$.
   a. $-2x(x+2)(x-3) < 0$
   b. $(x+4)(x+1)(x-2)^2 \leq 0$
2. Solve each of the following polynomial inequalities using an interval sign table, $x \in \mathbb{R}$.
   a. $2(x+3)(x-1)(x-5) \leq 0$
   b. $-3(x+4)(x-3)^3 > 0$
3. Solve the following polynomial inequalities, $x \in \mathbb{R}$.
   a. $x^2 - 4x + 3 < 0$
   b. $x^3 - 3x - 2 \geq 0$
   c. $x^4 - 1 \geq 0$
   d. $-x^2 + 3x + 1 < 0$
   e. $-2x^4 - 2x^3 + 16x^2 + 24x < 0$
4. Solve the following polynomial inequalities, $x \in \mathbb{R}$.
   a. $x^2 + x > 6$
   b. $x^3 < x$
   c. $x^3 + 2 \geq 2x^2 + x$
   d. $x^3 \geq x^2$
   e. $2x^2 - 2x \geq 2 - x$
   f. $x^4 < 22x^2 + 75$
5. An object is thrown into the air at a speed of v_0 m/s from a height of h_0 meters. The height h, in metres, of the object after t seconds is given by the equation
   $$h(t) = -4.9t^2 + v_0 \sin(\theta)t + h_0$$
   where θ is the angle between the object and the horizontal.
   A 1.8 metre tall quarterback throws the ball at a speed of 5.6 m/s to his receiver, at an angle of 30 degrees above horizontal. How high is the ball above the quarterback's head?
6. Let $f(x) = -2x + 1$, $g(x) = x^2 - 2x + 1$ and $h(x) = x^3 - 1$. Determine all values of x such that
   $$f(x) < g(x) < h(x)$$
   and illustrate the situation graphically.
7. The number n (in hundreds), of mosquitoes in a camping area after t weeks can be modelled by the equation
   $$n(t) = 2t^4 - 5t^3 - 16t^2 + 45t$$
   According to this model, when will the population of mosquitoes be greater than 1800?
8. A zoo wishes to construct an aquarium in the shape of a rectangular prism such that the length is twice the width and 5 m greater than the height. If the aquarium must have a volume strictly between 1125 m^3 and 3000 m^3, determine the restrictions on the length of the aquarium.
9. Determine the equation of a quintic function $f(x)$ that satisfies the following conditions:
   - $f(-3) = f(0) = f(4) = 0$
   - $f(1) = -9$
   - $f(x) > 0$ when $x < -3$ or $-3 < x < 0$
   - $f(x) < 0$ when $0 < x < 4$ or $x > 4$

   Illustrate the situation graphically.
10. The solution to $x^2 + bx + 24 < 0$ is the set of all values of x such that $k < x < k + 2$ for some real value of k. Determine all possible values of $b,\ b \in \mathbb{R}$. Justify your answer.
11. A quartic function has turning points at $(-3, 0)$, $(1, 0)$, and $(-1, -16)$. Determine all values of x such that $-9 < f(x) < 0$.

Solving Polynomial Inequalities

Partial Solutions

1. There is no solution provided for this question.

2. a. By inspection, a zero value occurs at $x = -3, 1$, and 5. This gives the following interval table:

	$x < -3$	$-3 < x < 1$	$1 < x < 5$	$x > 5$
$2(x+3)$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x-5$	$-$	$-$	$-$	$+$
$2(x+3)(x-1)(x-5)$	$-$	$+$	$-$	$+$

Therefore, $2(x+3)(x-1)(x-5) \le 0$ for $\{x \mid x \le -3 \text{ or } 1 \le x \le 5, x \in \mathbb{R}\}$.

b. By inspection, a zero value occurs at $x = -4$ and 3. This gives the following interval table:

	$x < -4$	$-4 < x < 3$	$x > 3$
$-3(x+4)$	$+$	$-$	$-$
$(x-3)^3$	$-$	$-$	$+$
$-3(x+4)(x-3)^3$	$-$	$+$	$-$

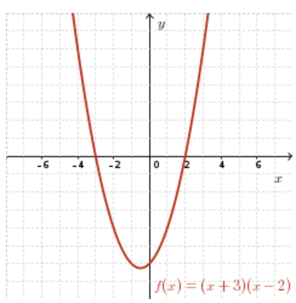
Therefore, $-3(x+4)(x-3)^3 > 0$ for $\{x \mid -4 < x < 3, x \in \mathbb{R}\}$.

3. There is no solution provided for this question.

4. a. Rearrange the inequality and factor.

$$x^2 + x > 6$$
$$x^2 + x - 6 > 0$$
$$(x+3)(x-2) > 0$$

Let $f(x) = (x+3)(x-2)$. Sketching the graph of $f(x)$,



we see that $f(x) > 0$ for $x < -3$ or $x > 2, x \in \mathbb{R}$.

b. Rearrange the inequality and factor.

$$x^3 < x$$
$$x^3 - x < 0$$
$$x(x^2 - 1) < 0$$
$$x(x-1)(x+1) < 0$$

By inspection, a zero value occurs at $x = -1, 0$ and 1. This gives the following interval table:

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
x	$-$	$-$	$+$	$+$

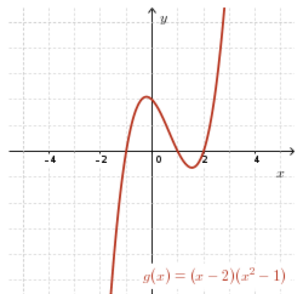
$x-1$	$-$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$	$+$
$x(x-1)(x+1)$	$-$	$+$	$-$	$+$

We see that $x(x-1)(x+1) < 0$ when $x < -1$ and $0 < x < 1, x \in \mathbb{R}$.

c. Rearrange the inequality and factor.

$$x^3 + 2 \geq 2x^2 + x$$
$$x^3 - 2x^2 - x + 2 \geq 0$$
$$(x-2)(x^2-1) \geq 0$$
$$(x-2)(x-1)(x+1) \geq 0$$

Let $g(x) = (x-2)(x-1)(x+1)$. Sketching the graph of $g(x)$,

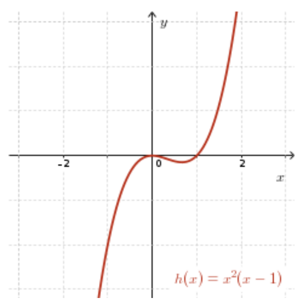


we see that $g(x) \geq 0$ for $-1 \leq x \leq 1$ or $x \geq 2, x \in \mathbb{R}$.

d. Rearrange the inequality and factor.

$$x^3 \geq x^2$$
$$x^3 - x^2 \geq 0$$
$$x^2(x-1) \geq 0$$

Let $h(x) = x^2(x-1)$. Sketching the graph of $h(x)$,

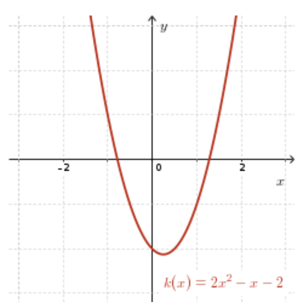


we see that $h(x) \geq 0$ for $x = 0$ or $x \geq 1, x \in \mathbb{R}$.

e. Rearrange the inequality to obtain

$$2x^2 - 2x \geq 2 - x$$
$$2x^2 - x - 2 \geq 0$$

The zeros of $2x^2 - x - 2 = 0$ are $\dfrac{1 \pm \sqrt{17}}{4}$. Let $k(x) = 2x^2 - x - 2$. Sketching the graph of $k(x)$,



we see that $k(x) \geq 0$ when $x \leq \dfrac{1-\sqrt{17}}{4}$ and $x \geq \dfrac{1+\sqrt{17}}{4}, x \in \mathbb{R}$.

f. Rearrange the inequality and factor.

$$x^4 < 22x^2 + 75$$
$$x^4 - 22x^2 - 75 < 0$$
$$(x^2 - 25)(x^2 + 3) < 0$$
$$(x - 5)(x + 5)(x^2 + 3) < 0$$

By inspection, a zero value occurs at $x = -5$ and 5. This gives the following interval table:

	$x < -5$	$-5 < x < 5$	$x > 5$
$x - 5$	$-$	$-$	$+$
$x + 5$	$-$	$+$	$+$
$x^2 + 3$	$+$	$+$	$+$
$(x - 5)(x + 5)(x^2 + 3)$	$+$	$-$	$+$

we see that $(x - 5)(x + 5)(x^2 + 3) < 0$ when $-5 < x < 5, x \in \mathbb{R}$.

5. There is no solution provided for this question.

6. First, solve $f(x) < g(x)$:

$$-2x + 1 < x^2 - 2x + 1$$
$$0 < x^2$$

Thus, the solution to $f(x) < g(x)$ is $\{x \mid x \neq 0, x \in \mathbb{R}\}$.

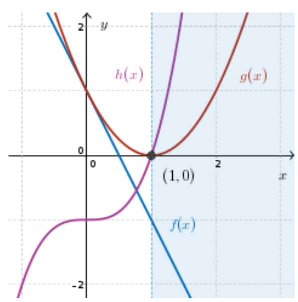
Then solve $g(x) < h(x)$:

$$x^2 - 2x + 1 < x^3 - 1$$
$$0 < x^3 - x^2 + 2x - 2$$
$$0 < x^2(x - 1) + 2(x - 1)$$
$$0 < (x - 1)(x^2 + 2)$$

Since $x^2 + 2 > 0$ for all x, the right hand side is positive whenever $x - 1 > 0$, that is, when $x > 1$. Thus, the solution to $g(x) < h(x)$ is $\{x \mid x > 1, x \in \mathbb{R}\}$.

Combining these two conditions, we see that the solution to $f(x) < g(x) < h(x)$ is $\{x \mid x > 1, x \in \mathbb{R}\}$.

Graphically, the solution to $f(x) < g(x) < h(x)$ is highlighted:



7. There is no solution provided for this question.

8. Let x represent the width of the aquarium ($x > 0$). Then its length is $2x$, its height is $2x - 5$, and the volume of the aquarium is

$$V(x) = x(2x)(2x - 5) = 4x^3 - 10x^2$$

We need to determine restrictions on x so that the volume $V(x)$ is between 1125 m^3 and 3000 m^3. That is, we are trying to solve the inequality

$$1125 < 4x^3 - 10x^2 < 3000$$

or equivalently, $1125 < 4x^3 - 10x^2$ and $4x^3 - 10x^2 < 3000$.

Solving $1125 < 4x^3 - 10x^2$,

$$1125 < 4x^3 - 10x^2$$
$$0 < 4x^3 - 10x^2 - 1125$$
$$0 < (2x - 15)(2x^2 + 10x + 75)$$

using the rational roots test and factor theorem. Since $x > 0$, $2x^2 + 10x + 75$ is always positive, the inequality holds when $2x - 15 > 0$, or $x > \frac{15}{2}$.

Solving $4x^3 - 10x^2 < 3000$,

$$4x^3 - 10x^2 < 3000$$
$$4x^3 - 10x^2 - 3000 < 0$$
$$2(x - 10)(2x^2 + 15x + 150) < 0$$

using the rational roots test and factor theorem. Since $2(2x^2 + 15x + 150)$ is always positive, the inequality holds when $x - 10 < 0$, or $x < 10$.

Combinining these restrictions, the solution to $1125 < V(x) < 3000$ is $\{x \mid \frac{15}{2} < x < 10, x \in \mathbb{R}\}$. Since the length is $2x$, we see that the length must be between 15 and 20 metres to have the desired volume.

9. There is no solution provided for this question.

10. Since the solution to $x^2 + bx + 24 < 0$ is the set of all values of x such that $k < x < k + 2$, the expression $x^2 + bx + 24$ must change sign at k and $k + 2$. Then $k, k + 2$ are the roots of the equation $x^2 + bx + 24 = 0$.
Using the general formula for quadratic equations,
$$x^2 - \left(\text{sum of roots}\right)x + \left(\text{product of roots}\right) = 0$$
the quadratic with roots k and $k + 2$ is given by $x^2 - (2k + 2)x + k(k + 2) = x^2 + bx + 24$, so

$$-(2k + 2) = b \quad \text{and} \quad \begin{aligned} k(k + 2) &= 24 \\ k^2 + 2k - 24 &= 0 \\ (k + 6)(k - 4) &= 0 \\ k &= -6 \text{ or } 4 \end{aligned}$$

When $k = -6$, $b = 10$.
When $k = 4$, $b = -10$.
Therefore, $b = -10$ or 10.

11. There is no solution provided for this question.